

A new lower bound for the chromatic number of general Kneser hypergraphs

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Abstract

A general Kneser hypergraph $KG^r(\mathcal{H})$ is an r -uniform hypergraph that somehow encodes the intersections of a ground hypergraph \mathcal{H} . The colorability defect of \mathcal{H} is a combinatorial parameter providing a lower bound for the chromatic number of $KG^r(\mathcal{H})$ which is involved in a series of works by Dol'nikov [Sibirskii Matematicheskii Zhurnal, 1988], Kříž [Transaction of the American Mathematical Society, 1992], and Ziegler [Inventiones Mathematicae, 2002]. In this paper, we define a new combinatorial parameter, the equitable colorability defect of hypergraphs, which provides some common improvements of these works. Roughly speaking, we propose a new lower bound for the chromatic number of general Kneser hypergraphs which substantially improves Ziegler's lower bound. It is always as good as Ziegler's lower bound and we provide several families of hypergraphs for which the difference between these two lower bounds is arbitrary large. This specializes to a substantial improvement of the Dol'nikov-Kříž lower bound for the chromatic number of general Kneser hypergraphs as well. Furthermore, we prove a result ensuring the existence of a colorful subhypergraph in any proper coloring of general Kneser hypergraphs which strengthens Meunier's result [The Electronic Journal of Combinatorics, 2014].

Keywords: general Kneser hypergraph, colorability defect, chromatic number, colorful subhypergraph.

1. Introduction and Main Results

For two positive integers n and k , two symbols $[n]$ and $\binom{[n]}{k}$ respectively stand for the set $\{1, \dots, n\}$ and the family of all k -subsets of $[n]$. Let n, k , and r be positive integers, where $r \geq 2$ and $n \geq rk$. The Kneser hypergraph $KG^r(n, k)$ is an r -uniform hypergraph with the vertex set $\binom{[n]}{k}$ and the edge set consisting of all r -tuples of pairwise disjoint members of $\binom{[n]}{k}$, i.e.,

$$E(KG^r(n, k)) = \{ \{e_1, \dots, e_r\} : |e_i| = k, e_i \subseteq [n] \text{ and } e_i \cap e_j = \emptyset \text{ for each } i \neq j \in [r] \}.$$

In a break-through, Lovász [14] determined the chromatic number of Kneser graphs $KG^2(n, k)$ and solved a long-standing conjecture posed by Kneser [11]. His proof gave birth to an area of combinatorics which nowadays is known as the topological combinatorics. As a main object, this area of combinatorics focuses on studying the coloring properties of graphs and hypergraphs by using algebraic topological tools. Alon, Frankl, and Lovász [4] extended Lovász's result to Kneser hypergraphs $KG^r(n, k)$ by proving

$$\chi(KG^r(n, k)) = \left\lceil \frac{n - r(k - 1)}{r - 1} \right\rceil.$$

This result gave an affirmative answer to a conjecture posed by Erdős [9] as well. Above-mentioned results are generalized in many ways. One of the most promising generalizations is the one found by Dol'nikov [8] and extended

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by Kříž [12, 13]. These results were also generalized by Ziegler [20, 21]. To state these results, we first need to introduce some preliminary notations and definitions.

Let $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ be a hypergraph and r be an integer, where $r \geq 2$. The general Kneser hypergraph $\text{KG}^r(\mathcal{H})$ is a hypergraph with the vertex set $E(\mathcal{H})$ and the edge set

$$E(\text{KG}^r(\mathcal{H})) = \{e_1, \dots, e_r : e_i \in E(\mathcal{H}) \text{ and } e_i \cap e_j = \emptyset \text{ for each } i \neq j \in [r]\}.$$

Note that the Kneser hypergraph $\text{KG}(n, k)$ can be obtained in this way by setting $\mathcal{H} = ([n], \binom{[n]}{k})$. The r -colorability defect of \mathcal{H} , denoted by $\text{cd}^r(\mathcal{H})$, is the minimum number of vertices that should be removed from \mathcal{H} so that the induced subhypergraphs on the remaining vertices has the chromatic number at most r . Dol'nikov [8] (for $r = 2$) and Kříž [12, 13] proved that

$$\chi(\text{KG}^r(\mathcal{H})) \geq \left\lceil \frac{\text{cd}^r(\mathcal{H})}{r-1} \right\rceil.$$

Note that if we set $\mathcal{H} = ([n], \binom{[n]}{k})$, then this result implies $\text{KG}^r(n, k) \geq \left\lceil \frac{n-r(k-1)}{r-1} \right\rceil$ bringing in Alon, Frankl, and Lovász's result [4] (note that one can simply color $\text{KG}^r(n, k)$ by $\left\lceil \frac{n-r(k-1)}{r-1} \right\rceil$ colors).

The equitable r -colorability defect of \mathcal{H} , denoted by $\text{ecd}^r(\mathcal{H})$, is the minimum number of vertices that should be removed from \mathcal{H} so that the induced subhypergraph on the remaining vertices admits an equitable r -coloring. We remind the reader that a hypergraph has an equitable r -coloring if it admits a proper r -coloring such that the sizes of its color classes differ by at most one. As an immediate consequence of one of our main results (Theorem 3), we have the following improvement of the Dol'nikov-Kříž lower bound.

Theorem 1. For any hypergraph \mathcal{H} and any integer r with $r \geq 2$, we have

$$\chi(\text{KG}^r(\mathcal{H})) \geq \left\lceil \frac{\text{ecd}^r(\mathcal{H})}{r-1} \right\rceil.$$

By the following example, we indicate that the previous theorem is a true improvement of the Dol'nikov-Kříž lower bound. More examples will be provided in the last section turning out that the difference between the preceding lower bound and the Dol'nikov-Kříž lower bound can be arbitrary large.

Example. Let n, k and a be integers, where $n \geq a \geq rk$. Define $\mathcal{H} = \mathcal{H}(n, k, a)$ to be a hypergraph with the vertex set $[n]$ and whose edge set is defined as follows;

$$E(\mathcal{H}) = \{A \subseteq [n] : |A| = k \text{ and } A \not\subseteq \{1, \dots, a\}\}.$$

In the last section of this paper, we shall prove that

$$\chi(\text{KG}^r(\mathcal{H})) = \left\lceil \frac{n - \max\{r(k-1), a\}}{r-1} \right\rceil.$$

In fact, we will see that the preceding result simply follows from the lower bound stated in Theorem 1. What is interesting about this example is that we cannot obtain the appropriate lower bound by using the colorability defect of \mathcal{H} (the Dol'nikov-Kříž lower bound) or even the alternation number of \mathcal{H} . It should be noticed that, for any hypergraph \mathcal{F} , there is a lower bound for the chromatic number of $\text{KG}^r(\mathcal{F})$ based on the alternation number of \mathcal{F} which surpasses the Dol'nikov-Kříž lower bound (the definitions and more details will be provided in Section 3.3). Some other examples comparing these lower bounds will be presented in Section 3.3 as well.

Let $\mathcal{H} = ([n], E(\mathcal{H}))$ be a hypergraph, r be an integer, and $s = (s_1, \dots, s_n)$ be an integer vector, where $r \geq 2$ and $1 \leq s_i < r$ for each $i \in [n]$. An r -(multi)set $\{N_1, \dots, N_r\}$ is called s -disjoint if each $i \in [n]$ appears in at most s_i number of N_j 's (we count the repetitions). Note that being $(1, \dots, 1)$ -disjoint is same as being pairwise disjoint. Ziegler [20] introduced the s -disjoint general Kneser hypergraph $\text{KG}_s^r(\mathcal{H})$ and the s -disjoint r -colorability defect $\text{cd}_s^r(\mathcal{H})$ as generalizations of the general Kneser hypergraph $\text{KG}^r(\mathcal{H})$ and the r -colorability defect $\text{cd}^r(\mathcal{H})$,

respectively. The s -disjoint general Kneser hypergraph $\text{KG}'_s(\mathcal{H})$ is an r -uniform (multi-)hypergraph with the vertex set $E(\mathcal{H})$ and the edge set

$$E(\text{KG}'_s(\mathcal{H})) = \{\{e_1, \dots, e_r\} : e_1, \dots, e_r \in E(\mathcal{H}) \text{ and } \{e_1, \dots, e_r\} \text{ is } s\text{-disjoint}\}.$$

Note that, depending on s , an edge of $\text{KG}'_s(\mathcal{H})$ is not necessarily a set and it might be a multi-set of size r . Also, the s -disjoint r -colorability defect of \mathcal{H} , denoted by $\text{cd}'_s(\mathcal{H})$, is defined as follows;

$$\text{cd}'_s(\mathcal{H}) = \bar{n} - \max \left\{ \sum_{j=1}^r |N_j| : N_j \subseteq [n], \{N_1, \dots, N_r\} \text{ is } s\text{-disjoint and } E(\mathcal{H}[N_j]) = \emptyset \text{ for each } j \in [r] \right\},$$

where $\bar{n} = \bar{n}(s) = \sum_{i=1}^n s_i$.

Ziegler [20, 21] generalized the Dol'nikov-Kříž lower bound to s -disjoint general Kneser hypergraphs. Indeed, he proved that

$$\chi(\text{KG}'_s(\mathcal{H})) \geq \left\lceil \frac{\text{cd}'_s(\mathcal{H})}{r-1} \right\rceil \quad (1)$$

provided that $\max_{i \in [n]} s_i < \mu(r)$, where $\mu(r)$ is the largest prime factor of r . Note that Ziegler's lower bound immediately implies the Dol'nikov-Kříž's lower bound since for $s = (1, \dots, 1)$, we have $\text{cd}'_s(\mathcal{H}) = \text{cd}^r(\mathcal{H})$ and $\text{KG}'_s(\mathcal{H}) = \text{KG}^r(\mathcal{H})$. Ziegler, using his lower bound, determined the chromatic number of the s -disjoint general Kneser hypergraph $\text{KG}'_s(\mathcal{H})$ for some integer vectors s .

An r -uniform hypergraph \mathcal{H} is called r -partite if its vertex set can be partitioned into subsets (parts) V_1, \dots, V_r so that each of its edges intersects each part V_i in exactly one vertex. An r -uniform r -partite hypergraph is said to be *complete* if it contains all possible edges. For the case $r = 2$, such a graph is called a *complete bipartite graph*.

Generalizing Lovász's result, several results concern the existence of colorful complete bipartite subgraphs in a properly colored G whose orders are related to some topological parameters associated to some topological spaces defined based on G , for instance see [2, 5, 6, 17, 18, 19]. In this regard, Simonyi and Tardos [19] proved that for an arbitrary hypergraph \mathcal{H} , any proper coloring of the general Kneser graph $\text{KG}(\mathcal{H})$ contains a multicolored complete bipartite subgraph $K_{\lfloor \frac{t}{2} \rfloor, \lceil \frac{t}{2} \rceil}$ of order $t = \text{cd}^2(\mathcal{H})$ in which the colors alternate on two parts of this bipartite subgraph with respect to their natural order. The existence of colorful subhypergraphs in hypergraphs coloring was first studied by Meunier [16]. Meunier generalized the Simonyi-Tardos result to general Kneser hypergraphs $\text{KG}^p(\mathcal{H})$ provided that p is prime. Actually, he proved that for a prime number p and a hypergraph \mathcal{H} , any coloring of $\text{KG}^p(\mathcal{H})$ contains a p -uniform p -partite subhypergraph with parts V_1, \dots, V_p satisfying the following properties;

- $\sum_{i=1}^p |V_i| = \text{cd}^p(\mathcal{H})$,
- the values of $|V_i|$'s differ by at most one, i.e., $\|V_i| - |V_j| \leq 1$ for each $i, j \in [p]$, and
- for any $j \in [p]$, the vertices in V_j get distinct colors.

This result was generalized to the family of all p -uniform hypergraphs in [1].

Let r and n be two integers and $s = (s_1, \dots, s_n)$ be an integer vector, where $r \geq 2$ and $1 \leq s_i < r$. For sets $N_1, \dots, N_r \subseteq [n]$, the (multi-)set $\{N_1, \dots, N_r\}$ is called *equitable* if the values of $|N_j|$'s differ by at most one. Also, it is called *equitable s -disjoint* if it is equitable and s -disjoint. For a hypergraph $\mathcal{H} = ([n], E(\mathcal{H}))$, define the *equitable s -disjoint r -colorability defect* of \mathcal{H} , denoted by $\text{ecd}'_s(\mathcal{H})$, to be the following quantity;

$$\text{ecd}'_s(\mathcal{H}) = \bar{n} - \max \left\{ \sum_{j=1}^r |N_j| : N_j \subseteq [n], \{N_1, \dots, N_r\} \text{ is equitable } s\text{-disjoint and } E(\mathcal{H}[N_j]) = \emptyset \text{ for each } j \in [r] \right\}.$$

We remind the reader that $\bar{n} = \sum_{i=1}^n s_i$. For $s = (1, \dots, 1)$, since $\text{ecd}^r(\mathcal{H})$ and $\text{ecd}'_s(\mathcal{H})$ are the same, we prefer to use $\text{ecd}^r(\mathcal{H})$ instead of $\text{ecd}'_s(\mathcal{H})$.

Next theorem is the first main result in the paper, which not only extends Meunier's result [16] to general s -disjoint Kneser hypergraphs $\text{KG}_s^p(\mathcal{H})$ but also improves it to equitable r -colorability defect. This theorem also results in a lower bound for the chromatic number of $\text{KG}_s^r(\mathcal{H})$ based on *equitable s -disjoint r -colorability defect* $\text{ecd}_s^r(\mathcal{H})$, which surpasses Ziegler's lower bound [20, 21].

Theorem 2. *Let $\mathcal{H} = ([n], E)$ be a hypergraph, p be a prime number, and $s = (s_1, \dots, s_n)$ be a positive integer vector, where $1 \leq s_i < p$ for each $i \in [n]$. Any proper coloring of the s -disjoint general Kneser hypergraph $\text{KG}_s^p(\mathcal{H})$ contains some subhypergraph whose vertex set can be partitioned into parts V_1, \dots, V_p satisfying the following properties;*

- $\sum_{i=1}^p |V_i| = \text{ecd}_s^p(\mathcal{H})$,
- $\{e_1, \dots, e_p\}$ forms an edge of $\text{KG}_s^p(\mathcal{H})$ for each choice of $e_i \in V_i$,
- the values of $|V_i|$'s differ by at most one, i.e., $\|V_i| - |V_j|\| \leq 1$ for each $i, j \in [p]$, and
- for any $j \in [p]$, the vertices in V_j get distinct colors.

Note that if we set $s = (1, \dots, 1)$, then the preceding theorem implies Meunier's theorem in a stronger form (using $\text{ecd}^r(\mathcal{H})$ instead of $\text{cd}^r(\mathcal{H})$). Also, since any color appears in at most $p - 1$ vertices of each edge of $\text{KG}_s^p(\mathcal{H})$, the previous theorem results in $\chi(\text{KG}_s^p(\mathcal{H})) \geq \left\lceil \frac{\text{ecd}_s^p(\mathcal{H})}{p - 1} \right\rceil$ provided that p is prime. The following theorem, applying this result, provides an improvement of Ziegler's theorem [20, 21].

Theorem 3. *Let $\mathcal{H} = ([n], E)$ be a hypergraph, r be an integer, $s = (s_1, \dots, s_n)$ be a positive integer vector, and $\mu(r)$ be the largest prime factor of r . If $1 \leq s_i < \mu(r)$ for each $i \in [n]$, then*

$$\chi(\text{KG}_s^r(\mathcal{H})) \geq \left\lceil \frac{\text{ecd}_s^r(\mathcal{H})}{r - 1} \right\rceil.$$

Since we always have $\text{ecd}_s^r(\mathcal{H}) \geq \text{cd}_s^r(\mathcal{H})$, setting $s = (1, \dots, 1)$ leads us to the improvement of the Dol'nikov-Kříž lower bound stated in Theorem 1.

Plan. The rest of this paper is organized as follows. In Section 2, we first introduce some topological tools which will be needed for the proofs of main results. Next, we prove Theorem 2 and then we deduce Theorem 3 from Theorem 2 by reducing Theorem 3 to the case of prime r . We end the paper in Section 2 by a discussion on comparing the equitable colorability defect with some other combinatorial parameters providing lower bounds for the chromatic number of general Kneser hypergraphs. In this section, we build some families of hypergraphs \mathcal{H} for which the difference between the lower bound introduced in Theorem 3 and some other well-known lower bounds for the chromatic number of general Kneser hypergraphs $\text{KG}^r(\mathcal{H})$ can be arbitrary large.

2. Proofs of the main results

This section is devoted to the proofs of Theorem 2 and Theorem 3. First, we review some basic definitions and tools. Next, we prove Theorem 2. To prove Theorem 3, we reduce this theorem to the case of prime r , which has been already proved by the discussion after Theorem 2.

2.1. Basic Tools

Here, a brief review of some notations and definitions, which will be used throughout this section, is provided though it is assumed that the reader has basic knowledge on topological combinatorics (see [15]).

For an integer r , by \mathbb{Z}_r , we mean the cyclic multiplicative group of order r and the generator ω , i.e., $\mathbb{Z}_r = \{\omega, \omega^2, \dots, \omega^r\}$. A *simplicial complex*, considered as a combinatorial object or a topological space, is a pair (V, K) , where V - the vertex set - is a finite set and K - the simplex set - is a hereditary system of nonempty subsets of V , that is, if $F \in K$ and $\emptyset \neq F' \subseteq F$, then $F' \in K$. By a simplicial complex K , we mean the simplicial complex (V, K) , where

$V = \bigcup_{F \in K} F$. Each set in K is called a *simplex* of K . The *dimension* of K , denoted by $\dim(K)$, equals $\max_{F \in K} |F| - 1$. The *first barycentric subdivision* of the simplicial complex K , denoted by $\text{sd } K$, is the order-complex obtained from the poset consisting of all simplices in K ordered by inclusion. The join of two simplicial complexes C and K , denoted by $C * K$, is a simplicial complex with the vertex set $V(C) \uplus V(K)$ and the simplex set $\{F_1 \uplus F_2 : F_1 \in C \text{ and } F_2 \in K\}$. For a prime number p and an integer s , where $1 \leq s < p$, the simplicial complex σ_{s-1}^{p-1} has \mathbb{Z}_p as the vertex set and its simplex set consists of all nonempty subsets of \mathbb{Z}_p with the size at most s . It is known that σ_{s-1}^{p-1} is a free $(s-2)$ -connected simplicial complex. Note that it implies that for any integer vector $(s_1, \dots, s_n) \in [p-1]^n$, the simplicial complex $\sigma_{s_1-1}^{p-1} * \dots * \sigma_{s_n-1}^{p-1}$ has the connectivity $\bar{n} - 2 = \sum_{i=1}^n s_i - 2$. Note that the vertex set of $\sigma_{s_1-1}^{p-1} * \dots * \sigma_{s_n-1}^{p-1}$ is $\mathbb{Z}_p \times [n]$ and its simplex set consists of all nonempty subsets A of $\mathbb{Z}_p \times [n]$ such that for each $i \in [n]$, the pair (ε, i) is appeared in A for at most s_i different $\varepsilon \in \mathbb{Z}_p$, i.e.,

$$\left| \{ \varepsilon \in \mathbb{Z}_p : (\varepsilon, i) \in A \} \right| \leq s_i.$$

Let C and K be two simplicial complexes. By a *simplicial map* $f : C \rightarrow K$, we mean a map from $V(C)$ to $V(K)$ such that the image of each simplex of C is a simplex of K . The simplicial complex C is a *simplicial \mathbb{Z}_p -complex* if \mathbb{Z}_p acts on it and moreover, the map $\varepsilon : C \rightarrow C$ which $v \mapsto \varepsilon \cdot v$ is a simplicial map for each $\varepsilon \in \mathbb{Z}_p$. A simplicial \mathbb{Z}_p -complex is free if there is no fixed simplex under the simplicial map made by each $\varepsilon \in \mathbb{Z}_p \setminus \{1\}$, where 1 is the identity element of the group \mathbb{Z}_p . For two simplicial \mathbb{Z}_p -complexes, a simplicial map $f : C \rightarrow K$ is called *\mathbb{Z}_p -equivariant* if $f(\varepsilon \cdot v) = \varepsilon \cdot f(v)$ for each $\varepsilon \in \mathbb{Z}_p$ and $v \in V(C)$. Simply, any \mathbb{Z}_p -equivariant simplicial map is called *simplicial \mathbb{Z}_p -map*.

2.2. Proof of Theorem 2

In our approach, we utilize the function $l(-)$ introduced by the second present author [1] and some sign functions proposed by Meunier [16]. Before starting the proof, we need to introduce these functions and some other notations.

Let n be a positive integer and p be a prime number. For a simplex $\tau \in (\sigma_{p-2}^{p-1})^{*n}$, define $h(\tau) = \min_{\varepsilon \in \mathbb{Z}_p} |\tau^\varepsilon|$, where $\tau^\varepsilon = \{(\varepsilon, j) : (\varepsilon, j) \in \tau\}$. Also, set

$$l(\tau) = p \cdot h(\tau) + |\{\varepsilon \in \mathbb{Z}_p : |\tau^\varepsilon| > h(\tau)\}|.$$

It should be emphasized that we shall use two functions $l(-)$ and $h(-)$ several times during the proof.

Proof. Let $\mathcal{H} = ([n], E)$ be a hypergraph, p be a prime number, and $s = (s_1, \dots, s_n)$ be a positive integer vector, where $1 \leq s_i < p$ for each $i \in [n]$. Also, let $c : E(\mathcal{H}) \rightarrow [t]$ be a proper coloring of $\text{KG}_s^p(\mathcal{H})$. When $\text{ecd}_s^r(\mathcal{H}) = 0$, the statement of Theorem 2 is clearly true. Therefore, for the rest of the proof, we assume that $\text{ecd}_s^r(\mathcal{H}) > 0$.

For a positive integer $a \in [n]$, let W_a be the set consisting of all simplices $\tau \in \sigma_{s_1-1}^{p-1} * \dots * \sigma_{s_n-1}^{p-1}$ such that $|\tau^\varepsilon| \in \{0, a\}$ for each $\varepsilon \in \mathbb{Z}_p$. Also, for a positive integer $b \in [t]$, let U_b be the set consisting of all simplices $\tau \in (\sigma_{p-2}^{p-1})^{*t}$ such that $|\tau^\varepsilon| \in \{0, b\}$ for each $\varepsilon \in \mathbb{Z}_p$. Define $W = \bigcup_{a=1}^n W_a$ and $U = \bigcup_{b=1}^t U_b$. Choose three arbitrary \mathbb{Z}_p -equivariant maps $s_0 : \sigma_{p-2}^{p-1} \rightarrow \mathbb{Z}_p$, $s_1 : W \rightarrow \mathbb{Z}_p$, and $s_2 : U \rightarrow \mathbb{Z}_p$ by choosing one representative in each orbit (note that this is possible because \mathbb{Z}_p acts freely on each of σ_{p-2}^{p-1} , W , and U).

Hereafter, for simplicity of notation, we set $K = \sigma_{s_1-1}^{p-1} * \dots * \sigma_{s_n-1}^{p-1}$. For each simplex τ of K and for each $\varepsilon \in \mathbb{Z}_p$, define

$$\tau_\varepsilon = \{j \in [n] : (\varepsilon, j) \in \tau\}.$$

Note that $\tau_\varepsilon \subseteq [n] = V(\mathcal{H})$. Furthermore, define

$$l(\mathcal{H}) = \max \{l(\tau) : \tau \in K \text{ such that } E(\mathcal{H}[\tau_\varepsilon]) = \emptyset \text{ for each } \varepsilon \in \mathbb{Z}_p\}.$$

One should notice that $\bar{n} - l(\mathcal{H}) = \text{ecd}_s^r(\mathcal{H})$. Now, set

$$m = l(\mathcal{H}) + \max \{l(\tau_c) : \tau \in K \text{ and } l(\tau) > l(\mathcal{H})\},$$

where, for each simplex τ in K ,

$$\tau_c = \{(\varepsilon, c(e)) \in \mathbb{Z}_p \times [t] : e \in E(\mathcal{H}) \text{ and } e \subseteq \tau_\varepsilon\}.$$

Note that for each simplex τ of K , the set $\{\tau_\varepsilon : \varepsilon \in \mathbb{Z}_p\}$ is s -disjoint. This observation alongs with the fact that c is a proper coloring of $\text{KG}_s^p(\mathcal{H})$ implies that τ_c is either empty or a simplex of $(\sigma_{p-2}^{p-1})^{*t}$. One should notice that this observation plays a crucial role in our proof.

In what follows, we first define a map $\gamma : \text{sd } K \longrightarrow \mathbb{Z}_p^{*m}$. Let τ be an arbitrary simplex in K . Define $\gamma(\tau)$ as follows.

a) If $l(\tau) \leq l(\mathcal{H})$. We consider two different cases.

1. If $h(\tau) = \min_{\varepsilon \in \mathbb{Z}_p} |\tau^\varepsilon| = 0$, then define $\gamma(\tau) = (s_0(\bar{\tau}), l(\tau))$, where

$$\bar{\tau} = \{\varepsilon \in \mathbb{Z}_p : \tau^\varepsilon = \emptyset\} \in \sigma_{p-2}^{p-1}.$$

2. If $h(\tau) > 0$, then define $\gamma(\tau) = (s_1(\bar{\tau}), l(\tau))$, where

$$\bar{\tau} = \bigcup_{\{\varepsilon \in \mathbb{Z}_p : |\tau^\varepsilon| = h(\tau)\}} \tau^\varepsilon \in W.$$

b) If $l(\tau) > l(\mathcal{H})$. Note that since $l(\tau) > l(\mathcal{H})$, the set τ_c is not empty and thus it is a simplex of $(\sigma_{p-2}^{p-1})^{*t}$. Now, we consider two different cases.

1. If $h(\tau_c) = \min_{\varepsilon \in \mathbb{Z}_p} |\tau_c^\varepsilon| = 0$, then define $\gamma(\tau) = (s_0(\bar{\tau}_c), l(\mathcal{H}) + l(\tau_c))$, where

$$\bar{\tau}_c = \{\varepsilon \in \mathbb{Z}_p : \tau_c^\varepsilon = \emptyset\} \in \sigma_{p-2}^{p-1}.$$

2. If $h(\tau_c) > 0$, then define $\gamma(\tau) = (s_2(\bar{\tau}_c), l(\mathcal{H}) + l(\tau_c))$, where

$$\bar{\tau}_c = \bigcup_{\{\varepsilon \in \mathbb{Z}_p : |\tau_c^\varepsilon| = h(\tau_c)\}} \tau_c^\varepsilon \in U.$$

Claim 1. *The map γ is a \mathbb{Z}_p -simplicial map from $\text{sd } K$ to \mathbb{Z}_p^{*m} .*

Proof. First note that all the functions $s_0(-)$, $s_1(-)$, and $s_2(-)$ are \mathbb{Z}_p -equivariant maps. Accordingly, it is clear that γ is a \mathbb{Z}_p -equivariant map as well. For a contradiction, suppose that γ is not a simplicial map. In view of simplices in $\text{sd } K$ and \mathbb{Z}_p^{*m} , it implies that there are two simplices τ and τ' in K such that $\tau \subsetneq \tau'$, $\gamma(\tau) = (\varepsilon_1, \beta)$, and $\gamma(\tau') = (\varepsilon_2, \beta)$, where $\varepsilon_1 \neq \varepsilon_2$ and $\beta \in [m]$. In view of the definition of γ , it is clear that it is not possible to have $l(\tau) \leq l(\mathcal{H})$ and $l(\tau') > l(\mathcal{H})$. Therefore, we have either $l(\tau) \leq l(\tau') \leq l(\mathcal{H})$ or $l(\tau') > l(\tau) > l(\mathcal{H})$, which will be discussed separately in the following.

I) $l(\tau) \leq l(\tau') \leq l(\mathcal{H})$. Clearly, in view of the definition of γ in this case, we should have $l(\tau) = l(\tau')$. We consider two following situations.

1. If $h(\tau) = h(\tau') = 0$, then since $\tau \subsetneq \tau'$ and

$$\varepsilon_1 = s_0(\{\varepsilon \in \mathbb{Z}_p : \tau^\varepsilon = \emptyset\}) \neq s_0(\{\varepsilon \in \mathbb{Z}_p : \tau'^\varepsilon = \emptyset\}) = \varepsilon_2,$$

we have $\{\varepsilon \in \mathbb{Z}_p : \tau'^\varepsilon = \emptyset\} \subsetneq \{\varepsilon \in \mathbb{Z}_p : \tau^\varepsilon = \emptyset\}$. This implies that

$$l(\tau') = p - |\{\varepsilon \in \mathbb{Z}_p : \tau'^\varepsilon = \emptyset\}| > p - |\{\varepsilon \in \mathbb{Z}_p : \tau^\varepsilon = \emptyset\}| = l(\tau),$$

a contradiction.

2. If $h(\tau') > 0$. We know that

$$l(\tau) = p \cdot h(\tau) + |\{\varepsilon \in \mathbb{Z}_p : |\tau^\varepsilon| > h(\tau)\}| \text{ and } l(\tau') = p \cdot h(\tau') + |\{\varepsilon \in \mathbb{Z}_p : |\tau'^\varepsilon| > h(\tau')\}|.$$

The facts $l(\tau) = l(\tau')$ and

$$\max \left\{ |\{\varepsilon \in \mathbb{Z}_p : |\tau^\varepsilon| > h(\tau)\}|, |\{\varepsilon \in \mathbb{Z}_p : |\tau'^\varepsilon| > h(\tau')\}| \right\} \leq p - 1$$

simply implies that $h = h(\tau) = h(\tau')$ and

$$|\{\varepsilon \in \mathbb{Z}_p : |\tau^\varepsilon| > h\}| = |\{\varepsilon \in \mathbb{Z}_p : |\tau'^\varepsilon| > h\}|.$$

In view of the following equations

$$\varepsilon_1 = s_1\left(\bigcup_{\{\varepsilon \in \mathbb{Z}_p : |\tau^\varepsilon| = h\}} \tau^\varepsilon\right) \neq s_1\left(\bigcup_{\{\varepsilon \in \mathbb{Z}_p : |\tau'^\varepsilon| = h\}} \tau'^\varepsilon\right) = \varepsilon_2,$$

we must have

$$\bigcup_{\{\varepsilon \in \mathbb{Z}_p : |\tau^\varepsilon| = h\}} \tau^\varepsilon \neq \bigcup_{\{\varepsilon \in \mathbb{Z}_p : |\tau'^\varepsilon| = h\}} \tau'^\varepsilon.$$

This observation and the facts that $\tau \subsetneq \tau'$ and $h = \min_{\varepsilon \in \mathbb{Z}_p} |\tau^\varepsilon| = \min_{\varepsilon \in \mathbb{Z}_p} |\tau'^\varepsilon|$ imply that

$$\{\varepsilon \in \mathbb{Z}_p : |\tau'^\varepsilon| = h\} \subsetneq \{\varepsilon \in \mathbb{Z}_p : |\tau^\varepsilon| = h\},$$

which is impossible.

II) $l(\tau) \geq l(\tau') > l(\mathcal{H})$. First note that both τ_c and τ'_c are simplices in $(\sigma_{p-2}^{p-1})^{*t}$. Clearly, in view of the definition of γ in this case, we should have $l(\tau_c) = l(\tau'_c)$. Similar to the previous case, we will deal with two different cases $h(\tau_c) = h(\tau'_c) = 0$ and $h(\tau'_c) > 0$.

1. If $h(\tau) = h(\tau') = 0$, then since $\tau_c \subsetneq \tau'_c$ and

$$\varepsilon_1 = s_0(\{\varepsilon \in \mathbb{Z}_p : \tau_c^\varepsilon = \emptyset\}) \neq s_0(\{\varepsilon \in \mathbb{Z}_p : \tau'^\varepsilon_c = \emptyset\}) = \varepsilon_2,$$

we have $\{\varepsilon \in \mathbb{Z}_p : \tau'^\varepsilon_c = \emptyset\} \subsetneq \{\varepsilon \in \mathbb{Z}_p : \tau_c^\varepsilon = \emptyset\}$. This implies that

$$l(\tau'_c) = p - |\{\varepsilon \in \mathbb{Z}_p : \tau'^\varepsilon_c = \emptyset\}| > p - |\{\varepsilon \in \mathbb{Z}_p : \tau_c^\varepsilon = \emptyset\}| = l(\tau_c),$$

a contradiction.

2. If $h(\tau'_c) > 0$. We know that

$$l(\tau_c) = p \cdot h(\tau_c) + |\{\varepsilon \in \mathbb{Z}_p : |\tau_c^\varepsilon| > h(\tau_c)\}| \text{ and } l(\tau'_c) = p \cdot h(\tau'_c) + |\{\varepsilon \in \mathbb{Z}_p : |\tau'^\varepsilon_c| > h(\tau'_c)\}|.$$

The facts $l(\tau_c) = l(\tau'_c)$ and

$$\max \left\{ |\{\varepsilon \in \mathbb{Z}_p : |\tau_c^\varepsilon| > h(\tau_c)\}|, |\{\varepsilon \in \mathbb{Z}_p : |\tau'^\varepsilon_c| > h(\tau'_c)\}| \right\} \leq p - 1$$

simply implies that $h = h(\tau_c) = h(\tau'_c)$ and

$$|\{\varepsilon \in \mathbb{Z}_p : |\tau_c^\varepsilon| > h\}| = |\{\varepsilon \in \mathbb{Z}_p : |\tau'^\varepsilon_c| > h\}|.$$

In view of the following equations

$$\varepsilon_1 = s_1\left(\bigcup_{\{\varepsilon \in \mathbb{Z}_p : |\tau_c^\varepsilon| = h\}} \tau_c^\varepsilon\right) \neq s_1\left(\bigcup_{\{\varepsilon \in \mathbb{Z}_p : |\tau'^\varepsilon_c| = h\}} \tau'^\varepsilon_c\right) = \varepsilon_2,$$

we must have

$$\bigcup_{\{\varepsilon \in \mathbb{Z}_p : |\tau_c^\varepsilon| = h\}} \tau_c^\varepsilon \neq \bigcup_{\{\varepsilon \in \mathbb{Z}_p : |\tau'_c| = h\}} \tau'_c.$$

This observation and the facts that $\tau_c \subsetneq \tau'_c$ and $h = \min_{\varepsilon \in \mathbb{Z}_p} |\tau_c^\varepsilon| = \min_{\varepsilon \in \mathbb{Z}_p} |\tau'_c|$ imply that

$$\{\varepsilon \in \mathbb{Z}_p : |\tau'_c| = h\} \subsetneq \{\varepsilon \in \mathbb{Z}_p : |\tau_c^\varepsilon| = h\},$$

which is impossible. ■

Claim 2. *There is a simplex $\tau \in K$ for which we have $l(\tau) > l(\mathcal{H})$ and $l(\tau_c) \geq \text{ecd}_s^p(\mathcal{H})$.*

Proof. It has already been noted that γ is a simplicial \mathbb{Z}_p -map from $\text{sd } K$ to \mathbb{Z}_p^{*m} , where

$$m = l(\mathcal{H}) + \max \{l(\tau_c) : \tau \in K \text{ and } l(\tau) > l(\mathcal{H})\}.$$

Accordingly, in view of Dold's theorem [7], the dimension of \mathbb{Z}_p^{*m} must be strictly larger than the connectivity of $\text{sd } K$; that is $m - 1 > \bar{n} - 2$, which implies $m \geq \bar{n}$. Consequently, in view of the definition of m , there is a simplex $\tau \in K$ for which we have $l(\tau) > l(\mathcal{H})$ and $l(\mathcal{H}) + l(\tau_c) \geq \bar{n}$. Equivalently, we have

$$l(\tau_c) \geq \bar{n} - l(\mathcal{H}) = \text{ecd}_s^p(\mathcal{H}),$$

as desired. ■

Let τ be a simplex for which we have $l(\tau) > l(\mathcal{H})$ and $l(\tau_c) \geq \text{ecd}_s^p(\mathcal{H})$. Set $h = \min_{\varepsilon \in \mathbb{Z}_p} |\tau_c^\varepsilon|$. For each $i \in [p]$, if $|\tau_c^{\omega^i}| = h$, then let $C_i = \{c_{i,1}, \dots, c_{i,h}\} \subseteq [t]$ where $\{\omega^i\} \times C_i = \tau_c^{\omega^i}$, otherwise, (if $|\tau_c^{\omega^i}| > h$) let $C_i = \{c_{i,1}, \dots, c_{i,h+1}\} \subseteq [t]$ be an $(h+1)$ -set such that $\{\omega^i\} \times C_i \subseteq \tau_c^{\omega^i}$. Now, for each $i \in [p]$, if $|\tau_c^{\omega^i}| = h$, then define $V_i = \{e_{i,1}, \dots, e_{i,h}\}$ such that $e_{i,j} \subseteq \tau_{\omega^i}$ and $c(e_{i,j}) = c_{i,j}$ for each $j \in [h]$, otherwise, define $V_i = \{e_{i,1}, \dots, e_{i,h+1}\}$ such that $e_{i,j} \subseteq \tau_{\omega^i}$ and $c(e_{i,j}) = c_{i,j}$ for each $j \in [h+1]$. It is clear that

$$\sum_{i=1}^p |V_i| = \sum_{i=1}^p |C_i| = l(\tau_c) \geq \text{ecd}_s^p(\mathcal{H}).$$

We have already noticed that the set $\{\tau_\varepsilon : \varepsilon \in \mathbb{Z}_p\}$ is s -disjoint, therefore, for each choice of $e_i \in V_i$, the set $\{e_1, \dots, e_p\}$ is s -disjoint as well and consequently $\{e_1, \dots, e_p\}$ is an edge of $\text{KG}_s^p(\mathcal{H})$. Now, one can simply see that the subgraph $\text{KG}_s^p(\mathcal{H})[\bigcup_{i=1}^p V_i]$ is the desired subhypergraph. ■

2.3. Proof of Theorem 3

In this subsection, we reduce Theorem 3 to the case of prime r , which is known to be true owing to Theorem 2. One should notice that the idea of this reduction is originally due to Kříž [13], which has been used in other papers as well, for instance see [2, 10, 20, 21].

For a hypergraph $\mathcal{H} = ([n], E)$ and positive integers r and C , define $\mathcal{T}_{\mathcal{H}, C, r}$ to be the hypergraph with the vertex set $V(\mathcal{H})$ and the edge set

$$E(\mathcal{T}_{\mathcal{H}, C, r}) = \{V \subseteq V(\mathcal{H}) : \text{ecd}^r(\mathcal{H}[V]) > (r-1)C\}.$$

Lemma 4. *Let r', r'' be two positive integers and $s = (s_1, \dots, s_n)$ be an integer vector, where $1 \leq s_i < r''$. For any hypergraph $\mathcal{H} = ([n], E)$, the following inequality holds*

$$\text{ecd}_s^{r' r''}(\mathcal{H}) \leq r''(r' - 1)C + \text{ecd}_s^{r''}(\mathcal{T}_{\mathcal{H}, C, r'}). \quad (2)$$

Proof. For simplicity of notations, set $\mathcal{T} = \mathcal{T}_{\mathcal{H},C,r'}$. Based on the definition of $\text{ecd}_s^{r''}(\mathcal{T})$, there exists a family of equitable s -disjoint subsets of $[n]$, say $\{V_1, \dots, V_{r''}\}$, such that $\sum_{i=1}^{r''} |V_i| = \sum_{i=1}^n s_i - \text{ecd}_s^{r''}(\mathcal{T})$ and $E(\mathcal{T}[V_i]) = \emptyset$ for each $i \in [r'']$. Clearly, this implies that $V_i \notin E(\mathcal{T})$ for each $i \in [r'']$ and consequently, $\text{ecd}^{r'}(\mathcal{H}[V_i]) \leq (r' - 1)C$ for each $i \in [r'']$. Thus, for each $i \in [r'']$, there is a family $\{V_{i1}, \dots, V_{ir'}\}$ of equitable disjoint subsets of V_i such that

$$\sum_{j=1}^{r'} |V_{ij}| = |V_i| - \text{ecd}^{r'}(\mathcal{H}[V_i]) \geq |V_i| - (r' - 1)C$$

and $E(\mathcal{H}[V_{ij}]) = \emptyset$ for all $j \in [r']$. One can simply check that the family $\{V_{ij} : 1 \leq i \leq r'' \text{ \& } 1 \leq j \leq r'\}$ is an equitable s -disjoint family of subsets of $[n]$ and moreover, $E(\mathcal{H}[V_{ij}]) = \emptyset$ for each $i \in [r'']$ and $j \in [r']$. Accordingly,

$$\begin{aligned} \text{ecd}_s^{r'r''}(\mathcal{H}) &\leq \sum_{i=1}^n s_i - \sum_{i=1}^{r''} \sum_{j=1}^{r'} |V_{ij}| \\ &\leq \sum_{i=1}^n s_i - \sum_{i=1}^{r''} |V_i| + r''(r' - 1)C \\ &= \text{ecd}_s^{r''}(\mathcal{T}) + r''(r' - 1)C \end{aligned}$$

as desired. ■

Lemma 5. Let r, r', r'' be positive integers, where $r', r'' \geq 2$ and $r = r'r''$. Also, let $s = (s_1, \dots, s_n)$ be positive integer vector, where $1 \leq s_i < r''$ for each $i \in [r'']$. If Theorem 3 holds for r' and $s_1 = (1, \dots, 1)$ and also for r'' and s , then it holds for r and s .

Proof. For sake a contradiction, suppose that there is a proper coloring $c : E(\mathcal{H}) \rightarrow [C]$ of $\text{KG}_s^r(\mathcal{H})$ for which $\text{ecd}_s^r(\mathcal{H}) > (r - 1)C$. Applying Lemma 4 leads us to

$$(r'r'' - 1)C < \text{ecd}_s^{r'r''}(\mathcal{H}) \leq r''(r' - 1)C + \text{ecd}_s^{r''}(\mathcal{T}_{\mathcal{H},C,r'}),$$

which immediately implies that $(r'' - 1)C < \text{ecd}_s^{r''}(\mathcal{T}_{\mathcal{H},C,r'})$. Note that since Theorem 3 holds for r'' and s , the preceding observation concludes in

$$\chi(\text{KG}_s^{r''}(\mathcal{T}_{\mathcal{H},C,r'})) > C.$$

On the other hand, in view of the definition of $\mathcal{T}_{\mathcal{H},C,r'}$, for each $e \in E(\mathcal{T}_{\mathcal{H},C,r'})$, we have $\text{ecd}^{r'}(\mathcal{H}[e]) > (r' - 1)C$. Since Theorem 2 holds for r' and $s_1 = (1, \dots, 1)$, this implies $\chi(\text{KG}^{r'}(\mathcal{H}[e])) > C$. Consequently the assignment of colors to the vertices of $\text{KG}^{r'}(\mathcal{H}[e])$ through the coloring c cannot be a proper coloring. Therefore, for each edge $e \in E(\mathcal{T}_{\mathcal{H},C,r'})$, there is at least one monochromatic edge in $\text{KG}^{r'}(\mathcal{H}[e])$. Now, for each edge $e \in E(\mathcal{T}_{\mathcal{H},C,r'})$, set $h(e)$ to be the largest color amongst the colors of monochromatic edges in $\text{KG}^{r'}(\mathcal{H}[e])$. Since c is a proper coloring of $\text{KG}_s^r(\mathcal{H})$, the map $h : E(\mathcal{H}) \rightarrow C$ is a proper coloring of $\text{KG}_s^{r''}(\mathcal{T}_{\mathcal{H},C,r'})$, which implies $\chi(\text{KG}_s^{r''}(\mathcal{T}_{\mathcal{H},C,r'})) \leq C$, a contradiction. To see this, contrary to the claim, suppose that $\{f_1, \dots, f_{r''}\}$ is an edge of $\text{KG}_s^{r''}(\mathcal{T}_{\mathcal{H},C,r'})$ such that $h(f_1) = \dots = h(f_{r''}) = a$. In view of the definition of $h(-)$, for each $j \in [r'']$, there is a monochromatic edge $\{e_{1,j}, \dots, e_{r',j}\}$ of $\text{KG}^{r'}(\mathcal{H}[f_j])$ for which we have $c(e_{1,j}) = \dots = c(e_{r',j}) = a$. One can simply see that $\{e_{i,j} : i \in [r'] \text{ \& } j \in [r'']\}$ is a monochromatic edge of $\text{KG}_s^r(\mathcal{H})$, contradicting the fact that c is a proper coloring for $\text{KG}_s^r(\mathcal{H})$. ■

In view of Lemma 5, to complete the proof of Theorem 3, it suffices to show that $\chi(\text{KG}_s^p(\mathcal{H})) \geq \left\lceil \frac{\text{ecd}_s^p(\mathcal{H})}{p-1} \right\rceil$ provided that p is prime. To this end, let c be a proper coloring of $\text{KG}_s^p(\mathcal{H})$ with color set $[C]$. Consider the subhypergraph whose existence is ensured by Theorem 2. Note that this subhypergraph has $\text{ecd}_s^p(\mathcal{H})$ vertices and any color appears in at most $p - 1$ vertices of each edge of $\text{KG}_s^p(\mathcal{H})$. This observation clearly results in $C \geq \left\lceil \frac{\text{ecd}_s^p(\mathcal{H})}{p-1} \right\rceil$, completing the proof of Theorem 3.

3. Comparing equitable colorability defect with colorability defect and alternation number

In this section, we compare equitable colorability defect of hypergraphs with colorability defect and alternation number of them, two other combinatorial parameters providing lower bounds for the chromatic number of general Kneser hypergraphs. It should be noticed that even though all examples presented in this section can be simply modified to the general case of s , for the ease of reading, we prefer to state them in the simpler case $s = (1, \dots, 1)$.

3.1. Alternation number

Let $\mathbb{Z}_r = \{\omega^j : j \in [r]\}$ be a cyclic and multiplicative group of order r with generator ω . For an $X = (x_1, \dots, x_n) \in (\mathbb{Z}_r \cup \{0\})^n$, the subsequence x_{i_1}, \dots, x_{i_m} of nonzero coordinates of X is called *alternating* if any two consecutive terms of this subsequence are different. In other words, the sequence x_{i_1}, \dots, x_{i_m} is called alternating whenever we have $1 \leq i_1 < \dots < i_m \leq n$, $x_{i_j} \neq 0$ for each $j \in [m]$ and $x_{i_j} \neq x_{i_{j+1}}$ for each $j \in [m-1]$. For each $X \in (\mathbb{Z}_r \cup \{0\})^n$, we define $\text{alt}(X)$ to be the longest alternating subsequence of X . Also, we define $\text{alt}(\mathbf{0}) = 0$. For each $X \in (\mathbb{Z}_r \cup \{0\})^n$ and for each $i \in [r]$, set $X^i = \{j \in [n] : x_j = \omega^i\}$. Note that by abuse of notation, we can write $X = (X^1, \dots, X^r)$.

Let \mathcal{H} be a hypergraph with the vertex set $V(\mathcal{H})$ and the edge set $E(\mathcal{H})$. Also, consider a bijection $\sigma : [n] \rightarrow V(\mathcal{H})$. The r -alternation number of \mathcal{H} with respect to the bijection σ , denoted by $\text{alt}'_\sigma(\mathcal{H})$, is the maximum possible k for which there is an $X \in (\mathbb{Z}_r \cup \{0\})^n$ with $\text{alt}(X) = k$ such that $E(\mathcal{H}[\sigma(X^i)]) = \emptyset$ for each $i \in [r]$, i.e.,

$$\text{alt}'_\sigma(\mathcal{H}) = \max \left\{ \text{alt}(X) : X \in (\mathbb{Z}_r \cup \{0\})^n \text{ and } E(\mathcal{H}[\sigma(X^i)]) = \emptyset \text{ for each } i \in [r] \right\}.$$

Now, define

$$\text{alt}'(\mathcal{H}) = \min_{\sigma} \text{alt}'_\sigma(\mathcal{H}),$$

where the maximum is taken over all bijections $\sigma : [n] \rightarrow V(\mathcal{H})$. The second present author and Hajiabolhassan [2], using Tucker's lemma, proved that

$$\chi(\text{KG}^r(\mathcal{H})) \geq \left\lceil \frac{|V(\mathcal{H})| - \text{alt}'(\mathcal{H})}{r-1} \right\rceil. \quad (3)$$

They also computed the chromatic number of several family of hypergraphs using this lower bound, see [2]. One can simply see that the preceding lower bound surpasses the Dol'nikov-Kříž lower bound.

3.2. Comparing equitable colorability defect with colorability defect

Based on the definition of $\text{cd}'_s(\mathcal{H})$ and $\text{ecd}'_s(\mathcal{H})$, it is clear that $\text{ecd}'_s(\mathcal{H}) \geq \text{cd}'_s(\mathcal{H})$. There are several examples ensuring that not only this inequality might be strict but also the difference between $\text{ecd}'_s(\mathcal{H})$ and $\text{cd}'_s(\mathcal{H})$ can be arbitrary large. To see this, as an instance, let \mathcal{H} be the complete bipartite graph $K_{t,t+c}$. Clearly, $\text{cd}^2(\mathcal{H}) = 0$ while $\text{ecd}^2(\mathcal{H}) = \min\{t, c-1\}$. This instance can be easily generalized to the complete r -partite hypergraph $\mathcal{H} = K_{t, \dots, t, T}$ where $T \geq rt + r - 1$. In this hypergraph, $\text{cd}^r(\mathcal{H}) = 0$ while $\text{ecd}^r(\mathcal{H}) = (r-1)t$ which provides a sharp lower bound for the chromatic number of $\text{KG}^r(\mathcal{H})$. Indeed, building a hypergraph \mathcal{H} for which $\text{ecd}^r(\mathcal{H}) - \text{cd}^r(\mathcal{H})$ is arbitrary large would be a very simple task. To see this, let m and n be positive integers, where $m \geq r(n+1)$. Now, consider an arbitrary hypergraph \mathcal{F} with n vertices and let $U = \{u_1, \dots, u_m\}$ be a set disjoint from $V(\mathcal{F})$. Define \mathcal{H} to be a hypergraph with the vertex set $V(\mathcal{F}) \cup U$ and the edge set $E(\mathcal{H}) = E(\mathcal{F}) \cup \{uv : u \in U \text{ and } v \in V\}$. It is clear that $\text{cd}^r(\mathcal{H}) = \text{cd}^{r-1}(\mathcal{F})$ and $\text{ecd}^r(\mathcal{H}) = n$, which results in $\text{ecd}^r(\mathcal{H}) - \text{cd}^r(\mathcal{H}) = n - \text{cd}^{r-1}(\mathcal{F})$. Now, note that we can consider several hypergraphs \mathcal{F} for which $n - \text{cd}^{r-1}(\mathcal{F})$ is arbitrary large.

3.3. Comparing equitable colorability defect with alternation number

In this section, we shall introduced some examples of hypergraphs \mathcal{H} for which we compare $\text{ecd}^r(\mathcal{H})$ with $\text{alt}'(\mathcal{H})$. Let n, r and k be positive integers, where $r, k \geq 2$ and $n \geq rk$. For a set $A \subsetneq [n]$, define $\mathcal{H}(n, k, A)$ be a hypergraph

with the vertex set $[n]$ and the edge set $E(\mathcal{H}) = \{e \in \binom{[n]}{k} : e \not\subseteq A\}$. It is simple to see that $\text{cd}^r(\mathcal{H}) = n - (r-1)(k-1) - \max\{|A|, k-1\}$. Also, one can check that the equitable colorability defect of \mathcal{H} is obtained by the following formula:

$$\text{ecd}^r(\mathcal{H}) = \begin{cases} n - r(k-1) & |A| \leq k-1 \\ n - r(k-1) - \left\lfloor \frac{|A|}{k} \right\rfloor & k \leq |A| \leq rk-2 \\ n - |A| & |A| \geq rk-1. \end{cases} \quad (4)$$

For computing the value of $\text{alt}^r(\mathcal{H})$, we need to put more efforts. Consider an arbitrary bijection $\sigma : [n] \rightarrow [n]$. If we set $X = (x_1, \dots, x_n)$ where

$$x_i = \begin{cases} \omega^i & i \in \{1, \dots, r(k-1)\} \\ 0 & \text{otherwise,} \end{cases}$$

then $\text{alt}(X) = r(k-1)$ and $E(\mathcal{H}[\sigma(X^i)]) = \emptyset$ for each $i \in [r]$, resulting in $\text{alt}_\sigma^r(\mathcal{H}) \geq r(k-1)$. In view of this observation, one can simply show that $\text{alt}^r(\mathcal{H}) = r(k-1)$ provided that $|A| \leq k-1$. Hereafter, we suppose that $|A| \geq k$. Also, note that without loss of generality and for simplicity of notation, we may assume that $A = \{1, \dots, a\}$. Let $\sigma : [n] \rightarrow [n]$ be an arbitrary fixed bijection. Also, let $\sigma^{-1}(A) = \{j_1, \dots, j_a\}$, where $1 \leq j_1 < \dots < j_a \leq n$. Three following different cases will be distinguished.

- **Case $r = 2$** : Define $X = (x_1, \dots, x_n)$ such that

$$x_i = \begin{cases} \omega & i = j_s \in \sigma^{-1}(A) \text{ and } s \text{ is even} \\ \omega^2 & i = j_s \in \sigma^{-1}(A) \text{ and } s \text{ is odd} \\ 0 & \text{otherwise,} \end{cases}$$

then clearly $\text{alt}(X) = |A|$ and $E(\mathcal{H}[\sigma(X^i)]) = \emptyset$ for each $i \in [r]$, implying that $\text{alt}_\sigma^r(\mathcal{H}) \geq |A|$. Therefore, since σ is arbitrary, we have $\text{alt}^r(\mathcal{H}) \geq |A|$.

- **Case $r = 3$** : Consider a fixed $i_0 \in [n] \setminus A$. Define $X = (x_1, \dots, x_n)$ such that

$$x_i = \begin{cases} \omega & i = j_s \in \sigma^{-1}(A) \text{ and } s \text{ is even} \\ \omega^2 & i = j_s \in \sigma^{-1}(A) \text{ and } s \text{ is odd} \\ \omega^3 & i = \sigma^{-1}(i_0) \\ 0 & \text{otherwise,} \end{cases}$$

then clearly $\text{alt}(X) = |A| + 1$ and $E(\mathcal{H}[\sigma(X^i)]) = \emptyset$ for each $i \in [r]$, implying that $\text{alt}_\sigma^r(\mathcal{H}) \geq |A| + 1$. Since σ is arbitrary, we have $\text{alt}^r(\mathcal{H}) \geq |A| + 1$.

- **Case $r \geq 4$** : Consider a fixed $B = \{a+1, \dots, a+b\} \subset [n] \setminus A$ such that $b = \min\{n - |A|, (r-2)(k-1)\}$. Also, let $\sigma^{-1}(B) = \{l_1, \dots, l_b\}$, where $1 \leq l_1 < \dots < l_b \leq n$. Define $X = (x_1, \dots, x_n)$ such that

$$x_i = \begin{cases} \omega & i = j_s \in \sigma^{-1}(A) \text{ and } s \text{ is even} \\ \omega^2 & i = j_s \in \sigma^{-1}(A) \text{ and } s \text{ is odd} \\ \omega^{3+t} & i = l_s \in \sigma^{-1}(B) \text{ and } s \equiv t \in \{0, \dots, r-3\} \pmod{r-2} \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\text{alt}(X) = |A| + b$ and $E(\mathcal{H}[\sigma(X^i)]) = \emptyset$ for each $i \in [r]$, implying that $\text{alt}_\sigma^r(\mathcal{H}) \geq |A| + b$. Since σ is arbitrary, we have $\text{alt}^r(\mathcal{H}) \geq |A| + b$.

By the following formula, we summarize the above-concluded observations.

$$n - \text{alt}^r(\mathcal{H}) \leq \begin{cases} n - \max\{2(k-1), |A|\} & r = 2 \\ n - \max\{3(k-1), |A| + 1\} & r = 3 \\ n - \max\{r(k-1), |A| + b\} & r \geq 4, \end{cases} \quad (5)$$

where $b = \min\{n - |A|, (r - 2)(k - 1)\}$.

In view of Formulas 4 and 5 and the fact that $\text{cd}^r(\mathcal{H}) = n - (r - 1)(k - 1) - \max\{|A|, k - 1\}$, if we define \mathcal{B} to be the family of hypergraphs $\mathcal{H}(n, k, A)$, then we clearly have the following observation.

Observation 6. *If $r \geq 4$, then for the hypergraphs $\mathcal{H} \in \mathcal{B}$, the values of $\text{ecd}^r(\mathcal{H}) - \text{cd}^r(\mathcal{H})$ and $\text{ecd}^r(\mathcal{H}) - (n - \text{alt}^r(\mathcal{H}))$ can be arbitrary large.*

For the hypergraph $\mathcal{H} = \mathcal{H}(n, k, A)$, the following proposition demonstrates that $\left\lceil \frac{\text{ecd}^r(\mathcal{H})}{r-1} \right\rceil$ provides an exact lower bound for the chromatic number of $\text{KG}^r(\mathcal{H})$ in some cases, while, in view of the previous observation, neither $\left\lceil \frac{\text{cd}^r(\mathcal{H})}{r-1} \right\rceil$ nor $\left\lceil \frac{n - \text{alt}^r(\mathcal{H})}{r-1} \right\rceil$ do that. Note that the following proposition can be considered as a generalization of the Alon-Frankl-Lovász theorem as well.

Proposition 7. *Let n, r , and k be positive integers, where $n \geq 2rk$. For $\mathcal{H} = \mathcal{H}(n, k, A)$, we have*

$$\chi(\text{KG}^r(\mathcal{H})) = \left\lceil \frac{n - \max\{r(k - 1), |A|\}}{r - 1} \right\rceil,$$

Provided that either $|A| \leq 2(k - 1)$ or $|A| \geq rk - 1$. Moreover, for $2k - 1 \leq |A| \leq rk - 2$, we have

$$\left\lceil \frac{n - r(k - 1) - \lfloor \frac{|A|}{k} \rfloor}{r - 1} \right\rceil \leq \chi(\text{KG}^r(\mathcal{H})) \leq \left\lceil \frac{n - \max\{r(k - 1), |A|\}}{r - 1} \right\rceil.$$

Proof. First, note that if $|A| < k$, then $\mathcal{H} = ([n], \binom{[n]}{k})$, which implies that $\chi(\text{KG}^r(\mathcal{H})) = \left\lceil \frac{n - r(k - 1)}{r - 1} \right\rceil$, in which each of $\text{cd}^r(\mathcal{H})$, $\text{ecd}^r(\mathcal{H})$, and $\text{alt}^r(\mathcal{H})$ provides the exact lower bound for the chromatic number of $\text{KG}^r(\mathcal{H})$. Therefore, we may assume that $|A| \geq k$. Clearly, the chromatic number of $\text{KG}^r(\mathcal{H})$ is bounded above by $\left\lceil \frac{n - r(k - 1)}{r - 1} \right\rceil$ since $\text{KG}^r(\mathcal{H})$ is a subhypergraph of $\text{KG}^r(n, k)$. Let (S_1, \dots, S_t) be a partition of $[n] \setminus A$, such that $|S_i| = r - 1$ for each $i \in \{1, \dots, t - 1\}$ and $1 \leq |S_t| \leq r - 1$. Note that $t = \left\lceil \frac{n - |A|}{r - 1} \right\rceil$. Consider the coloring $c : E(\mathcal{H}) \rightarrow [t]$ such that $c(e) = \min\{i \in [t] : e \cap S_i \neq \emptyset\}$. One can simply check that c is a proper coloring for $\text{KG}^r(\mathcal{H})$, resulting in

$$\chi(\text{KG}^r(\mathcal{H})) \leq \left\lceil \frac{n - \max\{r(k - 1), |A|\}}{r - 1} \right\rceil.$$

In what follows, considering three different cases, we introduce an appropriate lower bound for the chromatic number of $\chi(\text{KG}^r(\mathcal{H}))$ in each case, which completes the proof.

- For $k \leq |A| \leq 2(k - 1)$, it is trivial that $\text{alt}_I^r(\mathcal{H}) = r(k - 1)$, where $I : [n] \rightarrow [n]$ is the identity bijection. Therefore, in view of Inequality 3, we have the proof completed in this case.

- If $2k - 1 \leq |A| \leq rk - 2$, then $\text{ecd}^r(\mathcal{H})$ provides the lower bound $\left\lceil \frac{n - r(k - 1) - \lfloor \frac{|A|}{k} \rfloor}{r - 1} \right\rceil$ for $\chi(\text{KG}^r(\mathcal{H}))$ as desired.

- For $|A| \geq rk - 1$, the proof immediately follows by the fact that $\text{ecd}^r(\mathcal{H}) = n - |A|$.

■

Conjecture 8. *If $2k - 1 \leq |A| \leq rk - 1$, then*

$$\chi(\text{KG}^r(\mathcal{H})) = \left\lceil \frac{n - \max\{r(k - 1), |A|\}}{r - 1} \right\rceil.$$

This conjecture is strongly supported by Proposition 7. To be more specific, note that the validity of this conjecture is already verified for $r = 2$ by Proposition 7. Furthermore, one can see that for all the cases of $r, k, |A|$ and n for which

we have $\left\lceil \frac{n - r(k-1) - \lfloor \frac{|A|}{k} \rfloor}{r-1} \right\rceil = \left\lceil \frac{n - \max\{r(k-1), |A|\}}{r-1} \right\rceil$, Proposition 7 gives an affirmative answer to this conjecture.

Even more, if $|A| \leq r(k-1)$ and r is a power of 2, then we can deduce Conjecture 8 from a result by Alon, Drewnowski, and Łuczak [3]. For $s \geq 2$, a subset A of $[n]$ is called s stable if $s \leq |x - y| \leq n - s$ for each $x \neq y \in A$. The induced subhypergraph of $\text{KG}^r(n, k)$ by the set of all s -stable vertices is called the s -stable Kneser hypergraph $\text{KG}_s^r(n, k)$. Alon, Drewnowski, and Łuczak [3] proved that $\chi(\text{KG}_s^r(n, k)) = \chi(\text{KG}^r(n, k))$ provided that r is a power of 2. They also conjectured that this result is true for general r . One can simply see that the hypergraph $\text{KG}^r(\mathcal{H})$ in the statement of Conjecture 8 is a super-hypergraph of $\text{KG}_s^r(n, k)$ provided that $|A| \leq r(k-1)$, which implies that Conjecture 8 is true provided that $|A| \leq r(k-1)$ and r is a power of 2. In this point of view, we can consider this conjecture a weak version of the Alon-Drewnowski-Łuczak conjecture.

Though the lower bound of $\chi(\text{KG}^r(\mathcal{H}))$ which is based on $\text{ecd}^r(\mathcal{H})$ might be appreciably better than the lower bound based on $\text{alt}^r(\mathcal{H})$, in general we cannot decide which one is better. There are several examples in which the lower bound of $\chi(\text{KG}^r(\mathcal{H}))$ based on $\text{alt}^r(\mathcal{H})$ is much better than the lower bound gained by $\text{ecd}^r(\mathcal{H})$. As an instance, for positive integers n, r , and k , let $\mathcal{K} = \mathcal{K}_{n, \dots, n}$ be a graph with the vertex set $[rn]$ and edge set $E(\mathcal{K}) = \{e \in \binom{[rn]}{2} : e \not\subseteq \{1 + (t-1)n, \dots, tn\} \ \forall t \in [r]\}$. By the definition of the hypergraph \mathcal{K} , it is observed that $V(\mathcal{K})$ is partitioned into r disjoint sets of order n none of them containing any edge of \mathcal{K} . So, $\text{ecd}^r(\mathcal{K}) = 0$, while it can be verified that

$$rn - \text{alt}^r(\mathcal{K}) \geq rn - \text{alt}_I^r(\mathcal{K}) = \begin{cases} \frac{r}{2}n & r \text{ is even} \\ \lceil \frac{r}{2} \rceil n - 1 & r \text{ is odd,} \end{cases}$$

where $I : [rn] \rightarrow [rn]$ is the identity bijection.

Remark 9. Let r be a fixed integer, where $r \geq 2$. For a given hypergraph \mathcal{F} , it is notable that though $\left\lceil \frac{\text{ecd}^r(\mathcal{F})}{r-1} \right\rceil$ provides a lower bound for the chromatic number of $\text{KG}^r(\mathcal{F})$, computing $\text{ecd}^r(\mathcal{F})$ is an NP-hard problem.

The proof of this observation is almost the same as the proof of a result in [16] (see Proposition 6 in [16]). However, for the sake of completeness, we sketch the proof in the following.

Let r be a fixed integer, where $r \geq 2$. Also, let G be a given graph for which we want to compute the independence number $\alpha(G)$. Define \bar{G} to be a graph which is constructed by the union of r vertex-disjoint copies of G such that any two vertices from different copies are adjacent. It is not difficult to see that

$$\text{ecd}^r(\bar{G}) = r(|V(G)| - \alpha(G)).$$

This observation implies that computing of equitable colorability defect of graphs is at least as difficult as computing the independence number of graphs. Since computing the independence number of graphs is known as an NP-hard problem, we have the proof completed.

Acknowledgment

We would like to thank professor Frédéric Meunier for his comments that helped to improve the presentation of the paper.

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